

Obstacles to the Factorization of Linear Partial Differential Operators into Several Factors.

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Abstract

We consider algorithms for the factorization of linear partial differential operators. We introduce several new theoretical notions in order to simplify such considerations. We define an obstacle and a ring of obstacles to factorizations. We derive some interesting facts about the new objects, for instance, that they are invariant under gauge transformations. An important theorem for the construction of factoring algorithms is proved: a factorization is defined uniquely from a certain moment on.

For operators of orders three and two, obstacles are found explicitly.

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1 Introduction

We investigate the problem of the factorization of a linear partial differential operator over some field. The starting point is the algorithm of Grigoriev–Schwarz [1], which extends a factorization (into coprime factors) of the symbol of an operator to a factorization of this operator. At the first step of the algorithm, only the highest terms of the factors of a factorization are known. At every succeeding step, either we determine the next component in each factor, or we conclude that there is no such factorization. In the latter case we lose all of the information about the operator that we obtained implicitly during the execution of the algorithm. Here, we suggest that the information can be used, and introduce the notions of partial factorizations and common obstacles. The latter are exactly the invariants of Laplace for a second-order strictly hyperbolic operator [8].

We use partial factorizations to prove Theorem 3.10 below, which states that a factorization is uniquely defined from a certain moment on. The theorem of Grigoriev–Schwarz [1] is a particular case of this theorem.

Neither a common obstacle, nor its symbol is invariant or unique in general; some examples can be seen in [3]. However, we consider the factor ring of the ring of polynomials (corresponding to the ring of linear differential operators) modulo certain homogeneous ideal. We call it the ring of obstacles. The symbols of all common obstacles belong to the same class of this factor ring! We call the class the obstacle to factorizations. It is significant that the obstacle is invariant under gauge transformations. Others interesting properties are also derived.

We compute explicit formulas for common obstacles for bivariate second- and third-order operators.

It is important to mention that the theory has an application: it was used to obtain a full system of invariants for bivariate hyperbolic third-order operator [5].

The present work is an (independent) continuation of the work [6], where we consider factorizations into two factors.

2 Preliminaries

Let K be some field. Let $\Delta = \{\partial_1, \dots, \partial_n\}$ be commuting derivations acting on K . Consider the ring of linear differential operators

$$K[D] = K[D_1, \dots, D_n],$$

where D_1, \dots, D_n correspond to the derivations $\partial_1, \dots, \partial_n$ respectively.

The totality of all linear differential operators of orders $\leq i$ with defined left and right multiplication is a K -bimodule, which we denote by $K_{\leq i}$. Thus we have the filtration $\dots \supset K_{\leq i} \supset K_{\leq i-1} \supset \dots \supset K_{\leq 0}$. Consider the associate algebra

$$Smb_* = \sum_{i \geq 0} Smb_i, \quad Smb_i = K_{\leq i} / K_{\leq i-1}.$$

K -module Smb_* is a commutative K -algebra, which is isomorphic to the ring of polynomials $K[X] = K[X_1, \dots, X_n]$ in n variables. The image of the operator $L \in K[D]$ by the natural projection is some element Sym_L of $K[X]$. Actually, the symbol of an operator is a homogeneous polynomial corresponding to the sum of the highest terms.

We use the notation

$$D^{(i_1, \dots, i_n)} := D_1^{i_1} \dots D_n^{i_n},$$

and define the order as follows:

$$|D^{(i_1, \dots, i_n)}| = \text{ord}(D^{(i_1, \dots, i_n)}) := i_1 + \dots + i_n,$$

and in addition the order of the zero operator is $-\infty$.

For a homogeneous polynomial $S \in K[X]$ we define the operator $\widehat{S} \in K[D]$, which is the result of the substitution of the operator D_i for each variable X_i . If there is no danger of misunderstanding we use just S to denote the operator \widehat{S} . By $K_i[D]$ we denote the set of all operators in $K[D]$ of order i .

Now every operator $L \in K[D]$ can be written as

$$L = \sum_{|J| \leq d} a_J D^J = \sum_{i=0}^d L_i, \tag{1}$$

where $a_J \in K$, $J \in \mathbf{N}^n$, and L_i is the i th component of L .

3 Partial Factorizations

We start by generalizing several notions that were introduced for the case of factorizations into two factors in [6].

Definition 3.1. Let $L \in K[D]$ and suppose that its symbol has a decomposition $\text{Sym}_L = S_1 \dots S_k$. Then we say that the factorization

$$L = F_1 \circ \dots \circ F_k, \quad \text{where } \text{Sym}_{F_i} = S_i, \quad \forall i \in \{1, \dots, k\},$$

is of the factorization type $(S_1)(S_2) \dots (S_k)$.

Definition 3.2. Let for some operators L , for $F_i \in K[D]$, $i = 1, \dots, k$ and for some $t \in \{0, \dots, \text{ord}(L)\}$

$$\text{ord}(L - F_1 \circ \dots \circ F_k) < t \quad (2)$$

holds. Then we say that $F_1 \circ \dots \circ F_k$ is a *partial factorization* of order t of the operator L . If in addition $S_i = \text{Sym}_{F_i}$, $i = 1, \dots, k$ (so $\text{Sym}_L = S_1 \dots S_k$), then this partial factorization is of the factorization type $(S_1) \dots (S_k)$.

Remark 3.3. Every usual factorization of $L \in K[D]$ is a partial factorization of order 0.

Remark 3.4. Let $L \in K[D]$, $\text{ord}(L) = d$. Then for every factorization of the symbol $\text{Sym}_L = S_1 \dots S_k$ the corresponding composition of operators $\widehat{S}_1 \circ \dots \circ \widehat{S}_k$ is a partial factorization of order d .

Let $L \in K[D]$ and $F_1 \circ \dots \circ F_k$ be a partial factorization of order t . Note that the condition (2) still holds if we change any term whose order is less than or equal to $t - (d - d_j)$ in any factor F_j , $j \in \{1, \dots, k\}$. Thus we obtain new partial factorizations of order less than or equal t . Thus we introduce the following definition.

Definition 3.5. Let $L \in K[D]$, $\text{Sym}_L = S_1 \dots S_k$, $\text{ord}(S_i) = d_i$, $i = 1, \dots, k$ and

$$F_1 \circ \dots \circ F_k, \quad F'_1 \circ \dots \circ F'_k$$

be partial factorizations of orders t and t' respectively. Let $t' < t$, then $F'_1 \circ \dots \circ F'_k$ is an extension of $F_1 \circ \dots \circ F_k$ if

$$\text{ord}(F_i - F'_i) < t - (d - d_i), \quad \forall i \in \{1, \dots, k\}.$$

Example 3.6. Consider the fifth-order operator

$$L = (D_1^2 + D_2 + 1) \circ (D_1^2 D_2 + D_1 D_2 + D_1 + 1).$$

Compositions of the type

$$(D_1^2 + \dots) \circ (D_1^2 D_2 + \dots),$$

where ellipses mean arbitrarily chosen terms of lower orders, are partial factorizations of order 5. Their extensions are the following fourth-order partial factorizations of the type

$$(D_1^2 + D_2 + \dots) \circ (D_1^2 D_2 + D_1 D_2 + \dots).$$

Remark 3.7. Let $L \in K[D]$. Then $F_1 \circ \dots \circ F_k$ is a partial factorization of L of the type $(S_1) \dots (S_k)$ if and only if $F_1 \circ \dots \circ F_k$ is an extension of a partial factorization $S_1 \circ \dots \circ S_2$.

Now we formulate two easy to prove facts, which will be useful for the proof of a theorem below.

Proposition 3.8. *Let S_1, S_2, p be homogeneous polynomials, in an arbitrary number of variables, of orders d_1, d_2, s ($0 < s < d_1 + d_2$) respectively. Let, in addition, S_1 and S_2 be coprime. Then there exists at most one pair (u, v) of homogeneous polynomials u and v of orders $s - d_1$ and $s - d_2$ respectively, such that*

$$S_1 \cdot u + S_2 \cdot v = p.$$

The second fact is the generalization of Proposition 3.8 to the case of non-coprime polynomials.

Proposition 3.9. *Let S_1, S_2, p be homogeneous polynomials of orders d_1, d_2, s respectively and in an arbitrary number of variables. Consider the polynomial S_0 of order d_0 , which is the greatest common divisor of S_1 and S_2 , and $0 < s < d_1 + d_2 - d_0$. Then there exist at most one pair (u, v) of homogeneous polynomials u and v of orders $s - d_1$ and $s - d_2$ respectively, such that*

$$S_1 \cdot u + S_2 \cdot v = p. \quad (3)$$

For every factorization $S_1 \cdot S_2$ of the symbol, the corresponding composition of operators $\widehat{S}_1 \circ \widehat{S}_2$ is a partial factorization of the operator L . In the case of coprime S_1 and S_2 there exists at most one extension of this partial factorization to a factorization of L [1]. However, if there exists a nontrivial common divisor of S_1 and S_2 , then this is not necessarily the case. Consider, for example, the operator of Blumberg-Landau [2]:

$$L = D_x^3 + xD_x^2D_y + 2D_x^2 + (2x + 2)D_xD_y + D_x + (2 + x)D_y,$$

which is a frequently cited instance of an operator that has two factorizations into different numbers of irreducible factors (that is, factors that cannot be factored into factors of smaller orders):

$$L = (D_x + 1) \circ (D_x + 1) \circ (D_x + xD_y) = (D_x^2 + xD_xD_y + D_x + (2 + x)D_y) \circ (D_x + 1).$$

The same operator L (the symbol of L is $X^3 + xX^2Y$) has a whole family of factorizations into two factors with the symbols $S_1 = X$ and $S_2 = X(X + XY)$ respectively:

$$L = \left(D_x + 1 + \frac{1}{x + f_1(y)} \right) \circ \left(D_x^2 + xD_xD_y + \left(1 - \frac{1}{x + f_1(y)} \right) D_x + \left(x + 1 - \frac{x}{x + f_1(y)} \right) D_y \right),$$

where $f_1(y) \in K$ is a functional parameter. Thus, there is no uniqueness of factorization in this case. Nevertheless, even in the case of non-coprime symbols of factors we may formulate the following:

Theorem 3.10. *Let $L \in K[D]$, $\text{Sym}_L = S_1 \cdot S_2$, $\text{ord}(L) = d$, and let the greatest common divisor of S_1 and S_2 be a homogeneous polynomial S_0 of order s . Then for every $(d - d_0)$ th order partial factorization of the type $(S_1)(S_2)$, there is at most one extension to a complete factorization of L of the same type.*

To prove the theorem, it is enough to prove the following lemma.

Lemma 3.11. *Let $L \in K[D]$, $\text{Sym}_L = S_1 \cdot S_2$, $\text{ord}(L) = d$, and let the greatest common divisor of S_1 and S_2 be a homogeneous polynomial S_0 of order s . Then for every t -th ($t \leq (d - d_0)$) order partial factorization of the type $(S_1)(S_2)$, there is at most one (up to lower order terms) extension to the partial factorization of order $t - 1$ of the same type.*

Proof. If $d_0 = 0$, then the statement of the lemma is implied from [1]. If $d_0 > 0$, consider the general form of a complete factorization of L , that extends the given t -th order partial factorization:

$$L = \left(\widehat{S}_1 + \sum_{j=0}^{k_1-1} G_j \right) \circ \left(\widehat{S}_2 + \sum_{j=0}^{k_2-1} H_j \right), \quad (4)$$

where $k_1 = \text{ord}(S_1)$, $k_2 = \text{ord}(S_2)$ and $G_j \in K_j[D]$, $H_i \in K_i[D]$, $j = 0, \dots, (k_1 - 1)$, $i = 0, \dots, (k_2 - 1)$. By comparing components of order $t - 1$ on the both sides of the equality (4), we get

$$L_{t-1} = H_{t-k_1-1} \cdot S_1 + G_{t-k_2-1} \cdot S_2 + P_{t-1}, \quad (5)$$

where P_{t-1} is a homogeneous polynomial of order t , which is determined uniquely by the polynomials G_i, H_j , $i > t - k_1 - 1$, $j > t - k_2 - 1$, which are components of order t of the given partial factorization. We assume that polynomials G_i, H_i are 0 for $i < 0$.

Now, since $t - 1 < d - d_0$, we may apply proposition 3.9: there is at most one solution of equation (5). Thus there exists at most one extension to a partial factorization of order $t - 1$. \square

Corollary 3.12. *Let $L \in K[X]$, $\text{Sym}_L = S_1 \cdot S_2$, and S_1 and S_2 be coprime. Then there is at most one common factorization of L of the type $(S_1)(S_2)$. Thus the theorem is a generalization of the Grigoriev-Schwarz result [1].*

Corollary 3.13. *In the case of ordinary differential operators, the greatest common divisor of S_1 and S_2 is*

$$\gcd(S_1, S_2) = X^{d_0}, \quad \text{where } d_0 = \min(\text{ord}(S_1), \text{ord}(S_2)).$$

Then for every partial factorization of order

$$\max(\text{ord}(S_1), \text{ord}(S_2)) - 1$$

there is at most one extension to a complete factorization.

Corollary 3.14. *Let $L \in K[D]$, $\text{Sym}_L = S_1 \cdot S_2$, and let S_1 be coprime with S_2 . Then for every t , $t < \text{ord}(L)$ there is at most one (up to lower order terms) partial factorization of order t .*

4 Ring of Obstacles, Obstacles

By induction on the number of factors and with the theorem of uniqueness, if a factorization has the base [1], one may prove the following theorem:

Theorem 4.1. *Let $L \in K[D]$, $\text{Sym}_L = S_1 \cdot S_2 \dots S_k$ and let S_1, \dots, S_k be coprime. Then there exists at most one factorization of the type $(S_1)(S_2) \dots (S_k)$.*

Now, consider factorable operators as a subvariety of all the operators in $K[D]$ that have some fixed decomposition of the symbol.

Theorem 4.2. *Consider the variety of all the operators in $K[D]$ that have the symbol $\text{Sym} = S_1 \dots S_k$, $\text{ord}(S_i) = d_i$, $i = 1, \dots, k$. Then the codimension of the subvariety of the operators that have a factorization of the type $(S_1)(S_2) \dots (S_k)$ equals*

$$\binom{n+d-1}{n} - \sum_{i=1}^k \binom{n+d_i-1}{n}.$$

Proof. Consider the problem of the factorization of L of the type $(S_1)(S_2) \dots (S_k)$ in the general form:

$$L = \left(S_1 + \sum_{i=0}^{d_1-1} G_i^1 \right) \circ \dots \circ \left(S_k + \sum_{i=0}^{d_k-1} G_i^k \right), \quad (6)$$

where G_i^j denotes the i -th component in the j -th factor. Compare components of orders t , $0 \leq t \leq \text{ord}(L) - 1$ on both sides of (6), then we have

$$P_t = (\text{Sym}/S_1) \cdot G_{t-d+d_1}^1 + \dots + (\text{Sym}/S_k) \cdot G_{t-d+d_k}^k, \quad (7)$$

where P_t is a homogeneous polynomial of order t , which is determined uniquely by the polynomials G_i , H_j , $i > t - k_1$, $j > t - k_2$, and so it is known if we solve equations (7) in “descent” order, that is if we start with $t = \text{ord}(L) - 1$, and reduce t by one at each succeeding step.

Polynomials G_i, H_j , $i > t - k_1$, $j > t - k_2$, and so P_t are determined uniquely: it is an immediate consequence of the following lemma:

Lemma 4.3. *Let S_1, \dots, S_k are pairwise coprime homogeneous polynomials of orders d_1, \dots, d_k respectively. Denote $S = S_1 \dots S_k$. Then there is at most one tuple (A_1, \dots, A_k) such that*

$$P_t = (S/S_1) \cdot A_1 + \dots + (S/S_k) \cdot A_k, \quad (8)$$

where $\text{ord}(P_t) = t$, $t < \text{ord}(S)$, and $\text{ord}(A_i) + \text{ord}(S/S_i) = t$.

Proof. Assume we have two such tuples: (A'_1, \dots, A'_k) and (A''_1, \dots, A''_k) . Consider the difference of the equations corresponding to them, so we have

$$0 = (S/S_1) \cdot B_1 + \dots + (S/S_k) \cdot B_k, \quad (9)$$

where $B_i = A'_i - A''_i$, $i = 1, \dots, k$. Without loss of generality we may assume $B_1 \neq 0$ and rewrite equation (9) in the form

$$-(S/S_1) \cdot B_1 = (S/S_2) \cdot B_2 + \dots + (S/S_k) \cdot B_k.$$

Every component on the right is divisible by S_1 , while (S/S_1) is not so. Thus, B_1 is divisible by S_1 , and so $\text{ord}(B_1) \geq \text{ord}(S_1)$.

On the other hand, we have $\text{ord}(A_i) + \text{ord}(S/S_i) = t$ and $t < \text{ord}(S)$, that is $\text{ord}(A_i) < \text{ord}(S_i)$, and so $\text{ord}(B_1) < \text{ord}(S_i)$. This is a contradiction with the results of the previous paragraph. \square

The factorization exists if the system of all the equations (7), $t = d - 1, \dots, 0$ is compatible. The codimension equals the number of independent equations in the coefficients of the operator.

For every t we have the linear equation (7) in the polynomials $G_{t-d+d_1}^1, \dots, G_{t-d+d_k}^k$, which is equivalent to the system of linear equations in their coefficients. Let the system be $A \cdot \vec{g} = \vec{c}$, where A is the matrix of the system. The system has a unique solution, and so the rank of the matrix A equals the number v of variables. That is the columns of the matrix A are linearly independent.

The system $A \cdot \vec{g} = \vec{c}$ is compatible when vector \vec{c} belongs to a v -dimensional affine space, generated by the columns of A . The length of vector \vec{c} equals the number of equations in the system. Thus the codimension of the solution space is the difference between the number of equations and the number of variables.

Now the codimension of the variety of all the operators that have factorizations of the type $(S_1)(S_2) \dots (S_k)$ equals the difference between the number of equations and the number of variables at all the steps together. This can be computed using the following combinatorial fact:

Lemma 4.4. *The cardinality of the set*

$$\{M = x_1^{d_1} \dots x_n^{d_n} \mid d_1 + \dots + d_n = t\}$$

of monomials in n independent variables x_1, \dots, x_n is $\binom{n+t-1}{t} = \binom{n+t-1}{n-1}$.

The theorem about codimension is proved. \square

Example 4.5. Consider all the operators of order two in two independent variables with symbol $S_1 \cdot S_2$, where S_1, S_2 are coprime homogeneous operators of the first order. By Theorem 4.2, the codimension of the variety of all the operators that have a factorization of the type $(S_1)(S_2)$, is 1.

One may find explicit formulae for the equation which defines this variety. Let, for example, $\widehat{S}_1 = D_1$, $\widehat{S}_2 = D_2$. Consider all the operators of the form $L = D_1 D_2 + a_{10} D_1 + a_{01} D_2 + a_{00}$. Such an operator has a factorization of the type $(S_1)(S_2)$ if and only if coefficients a_{10}, a_{01}, a_{00} satisfy the condition

$$a_{00} - a_{10}a_{01} - \partial_x(a_{10}) = 0.$$

Example 4.6. Consider all the operators of order three in two independent variables with symbol $S_1 \cdot S_2$, where S_1, S_2 are coprime homogeneous operators of first and second orders respectively. By Theorem 4.2, the codimension of the variety of all the operators that have a factorization of the type $(S_1)(S_2)$, is 2.

However, if we consider a factorization of the type $(S_1)(S_2)(S_3)$, where S_1, S_2, S_3 are coprime homogeneous operators of the first order, then, by Theorem 4.2, the codimension is 3.

To study operators that have no factorization of some type, we introduce the following notion:

Definition 4.7. Let $L \in K[D]$, $\text{Sym}_L = S_1 \dots S_k$. An operator $R \in K[D]$ is called a *common obstacle* to factorization of the type $(S_1)(S_2) \dots (S_k)$ if there exists a factorization of this type for the operator $L - R$ and R has minimal possible order.

Common obstacles are closely related to partial factorizations:

Proposition 4.8. *Let $L \in K[D]$, $\text{Sym}_L = S_1 \dots S_k$. A common obstacle to a factorization of the type $(S_1) \dots (S_k)$ is of order t if and only if the minimal order of a partial factorization of this type is $t + 1$.*

Common obstacles and their symbols are not unique in general, and neither of them is invariant or has some interesting properties. That is why we introduce the following notion.

Definition 4.9. Let $L \in K[D]$ and $\text{Sym}_L = S_1 \cdot S_2 \cdot \dots \cdot S_k$. Then we say that the *ring of obstacles* to factorizations of the type $(S_1) \dots (S_k)$ is the factor ring

$$K(S_1, \dots, S_k) = K[X]/I,$$

where

$$I = \left(\frac{\text{Sym}_L}{S_1}, \dots, \frac{\text{Sym}_L}{S_k} \right)$$

is a homogeneous ideal.

Remark 4.10. In the case of two factors ($k = 2$), the ring of obstacles is

$$K(S_1, S_2) = K[X]/(S_1, S_2).$$

So the definition 4.9 is a generalization of the definition given in [6], where we study the case of factorizations into two factors.

Theorem 4.11. *Let $L \in K[D]$ and $\text{Sym}_L = S_1 \cdot S_2 \dots S_k$, where S_i , $i \in \{1, \dots, k\}$ are pairwise coprime. Then the symbols of all common obstacles to factorization of the type $(S_1) \dots (S_k)$ belong to the same class in the factor-ring $K(S_1, \dots, S_k)$.*

Proof. Denote $d_i = \text{ord}(S_i)$, $i \in \{1, \dots, k\}$ and let t be the order of common obstacles. In the same way as in the proof of Theorem 4.2, we obtain the equation (7), that is the symbol of every common obstacle can be written in the form

$$P_t - ((\text{Sym}_L/S_1) \cdot G_{t-d+d_1}^1 + \dots + (\text{Sym}_L/S_k) \cdot G_{t-d+d_k}^k),$$

where P_t is known, uniquely determined and the same for all common obstacles polynomial. Thus all common obstacles belong to the class $[P_t]$ of the factor-ring $K(S_1, \dots, S_k)$. \square

Definition 4.12. We say that the class of common obstacles in the ring of obstacles is the *obstacle to factorization*.

Remark 4.13. Every element of this class is again a common obstacle.

Definition 4.14. We say that two types of factorizations $(S_1) \dots (S_k)$ and $(b_1 S_1) \dots (b_k S_k)$ are *similar*, if $b_1, \dots, b_k \in K$ and $b_1 \dots b_k = 1$.

Theorem 4.15. *For an operator in $K[X]$ the rings of obstacles and the obstacles of similar types are the same.*

Proof. Consider an operator $L \in K[D]$ and two similar types of factorizations of L : $(S_1) \dots (S_k)$ and $(b_1 S_1) \dots (b_k S_k)$, where $b_i \in K$, $i = 1, \dots, k$. Then the homogeneous ideals (S_1, \dots, S_k) and $(b_1 S_1, \dots, b_k S_k)$ are the same, thus the rings of obstacles are also.

Every common obstacle of the type $(S_1) \dots (S_k)$ and of order d_0 may be written as

$$P = L - (\widehat{S}_1 + T_1) \circ \dots \circ (\widehat{S}_k + T_k), \quad (10)$$

where T_i is the sum of components of orders $d_i - 1, \dots, d - d_i - d_0 + 1$, and $\text{ord}(P) = d_0$.

There exist T'_1, \dots, T'_k such that T'_i is the sum of components of orders $d_i - 1, \dots, d - d_i - d_0 + 1$ and

$$(S_1 + T_1) \circ \dots \circ (S_k + T_k) = (b_1 S_1 + T'_1) \circ \dots \circ (b_k S_k + T'_k).$$

Thus P is a common obstacle of order d_0 of the type $(b_1 S_1) \dots (b_k S_k)$. On the other hand, we know that the rings of obstacles $K(S_1, \dots, S_k)$ and $(b_1 S_1, \dots, b_k S_k)$ are the same. Thus obstacles are the same also. \square

Let us recall the definition:

Definition 4.16. A gauge transformation of $L \in K[D]$ with an invertible element in $g \in K$ is the operator $g^{-1} \circ L \circ g$.

Theorem 4.17. Let P be a common obstacle for $L \in K[D]$, then $g^{-1} Pg$ will be a common obstacle for the gauge transformed operator $g^{-1} L g$, where $g \in K^*$ (K^* - the set of invertible elements in K).

Proof. Consider a common obstacle (10) for L of order d_0 . Then we have

$$g^{-1} Pg = g^{-1} L g - g^{-1} \circ (S_1 + T_1) \circ \Pi_{j=2}^{k-1} (S_j + T_j) \circ (S_k + T_k) \circ g.$$

There exist T'_1, \dots, T'_k such that T'_i is the sum of components of orders $d_i - 1, \dots, d - d_i - d_0 + 1$ and

$$g^{-1} Pg = g^{-1} L g - (g^{-1} S_1 + T'_1) \circ \Pi_{j=2}^{k-1} (S_j + T'_j) \circ (g S_k + T'_k).$$

\square

Corollary 4.18. Obstacles are invariant under the gauge transformations.

Proof. Under the gauge transformations common obstacles are conjugated, and so symbols of common obstacles are the same. \square

Theorem 4.19. Let $n = 2$, $L \in K[D]$, $\text{ord}(L) = d$, and let $\text{Sym}_L = S_1 \dots S_k$, where S_i , $i \in \{1, \dots, k\}$ are pairwise coprime. Thus the ring of obstacles $K(S_1, \dots, S_k)$ is 0 to order $d - 1$. (That is, non-zero obstacles may be only less than or equal to $d - 2$.)

Proof. Denote $d_i = \text{ord}(S_i)$, $i \in \{1, \dots, k\}$ and repeat the reasoning of the proof of Theorem 4.2. Thus we write equation (7) for $t = d - 1$:

$$P_{d-1} = (\text{Sym}_L / S_1) \cdot G_{d_1-1}^1 + \dots + (\text{Sym}_L / S_k) \cdot G_{d_k-1}^k.$$

It has at most one solution w.r.t. $G_{d_1-1}^1, \dots, G_{d_k-1}^k$. Consider the corresponding system of equations in their coefficients. By Lemma 4.4 the number of equations in this system is d , the number of variables is d also. Thus the system has a unique solution, and so we have a partial factorization of order $d - 1$. \square

Recall that an operator $L \in K[D]$, $\text{ord}(L) = d$ is called strictly hyperbolic if the symbol of L has exactly d different factors.

Theorem 4.20. *Let $n = 2$ and $L \in K[D]$ be strictly hyperbolic of order d . Then for each type of factorization, a common obstacle is unique.*

Proof. Let the type of the factorizations be $(S_1) \dots (S_d)$, and let P be a common obstacle for this type. Let the order of common obstacles be p . Assume there is another common obstacle for this type, then it is of the form

$$P + (\text{Sym}_L/S_1) \cdot A_1 + \dots + (\text{Sym}_L/S_d) \cdot A_d,$$

where A_i are some homogeneous polynomials of orders $p_i = p - \text{ord}(\text{Sym}_L/S_i) = p - (d - 1)$. That is $p \geq d - 1$.

On the other hand, by Theorem 4.19, the ring of obstacles is 0 to order $d - 1$, and so $p \leq d - 2$. \square

5 Bivariate Operators of Order Two

Consider a second-order hyperbolic operator $L \in K[D_x, D_y]$ that is in such a system of coordinate that the symbol of L is XY . Then by Theorems 4.19 and 4.20, both common obstacles to factorizations of L have order 0 and are uniquely defined. We compute explicit formulas.

Theorem 5.1. *Let*

$$L = D_x \cdot D_y + aD_x + bD_y + c,$$

where $a_{10}, a_{01}, a_{00} \in K$. Then obstacles of types $(X)(Y)$, $(Y)(X)$ are

$$\begin{aligned} c - ab - \partial_x(a), \\ c - ab - \partial_y(b) \end{aligned}$$

respectively.

Proof. A factorization of L of type $(X)(Y)$ has the form

$$L = (D_x + g_{00}) \circ (D_y + h_{00}),$$

where g_{00}, h_{00} are some elements of K . Comparing components of order 1 on the right and on the left, we have

$$(a - h_{00})D_x + (b - g_{00})D_y = 0, \quad (11)$$

that is $a = h_{00}$, $b = g_{00}$. Now we compute the obstacle as

$$L - (D_x + b) \circ (D_y + a) = c - ab - \partial_x(a).$$

One may find the obstacle for type $(Y)(X)$ analogously. \square

Remark 5.2. The obtained obstacles are the invariants of Laplace [8].

6 Bivariate Operators of Order Three

Consider some operator $L \in K[D_1, D_2]$ of order three. Let the symbol of L be $S_1 \cdot S_2 \cdot S_3$, then the following types of factorizations are possible: six types of factorization into three factors:

$$(S_1)(S_2)(S_3), (S_1)(S_3)(S_2), (S_2)(S_1)(S_3), (S_2)(S_3)(S_1), (S_3)(S_1)(S_2), (S_3)(S_2)(S_1),$$

and six types of factorization into two factors:

$$(S_1)(S_2S_3), (S_2)(S_1S_3), (S_3)(S_1S_2), (S_1S_2)(S_3), (S_1S_3)(S_2), (S_2S_3)(S_1).$$

6.1 Two Factors

The theory introduced above applies for the case of pairwise coprime symbols of factors. That is, if the considered type is $(S_1)(S_2S_3)$, then S_1 and S_2S_3 should be coprime. Taking this and the symmetry into account, we restrict ourselves to considering two important special cases: factorization of the type $(X)(X^2 + XY)$ for an operator with symbol $X^2Y + XY^2$ and of the type $(X)(Y^2)$ for an operator with symbol XY^2 .

Note that by Theorem 4.19 common obstacles of these types may be of orders one and zero only, in the first case a common obstacle is not unique.

Theorem 6.1. *Let*

$$L = \widehat{\text{Sym}}_L + a_{20}D_{xx} + a_{11}D_{xy} + a_{02}D_{yy} + a_{10}D_x + a_{01}D_y + a_{00},$$

where all $a_{ij} \in K$.

Let $\text{Sym}_L = XY(X + Y)$, then

$$\begin{aligned} \text{Obst}_{(X)(YX+YY)} = & \left(a_{02}^2 - a_{11}a_{02} + a_{01} + \partial_x(a_{02} - a_{11}) \right) D_y + \\ & a_{00} - a_{02}a_{10} + a_{02}^2a_{20} + 2a_{02}\partial_x(a_{20}) - \partial_x(a_{10}) + a_{20}\partial_x(a_{02}) + \partial_{xx}(a_{20}), \end{aligned}$$

is a common obstacle to factorizations of L of type $(X)(YX + YY)$.

Let $\text{Sym}_L = X^2Y$, then

$$\begin{aligned} \text{Obst}_{(Y)(XX)} = & \left(a_{10} - a_{20}a_{11} - \partial_y(a_{11}) \right) D_x + \\ & a_{00} - a_{20}a_{01} + a_{20}^2a_{02} + 2a_{20}\partial_y(a_{02}) - \partial_y(a_{01}) + a_{02}\partial_y(a_{20}) + \partial_{yy}(a_{02}), \end{aligned}$$

is a common obstacle to factorizations of L of type $(Y)(XX)$.

Proof. All factorizations of type $(X)(YX + YY)$ have the form

$$L = (D_x + G_0) \circ (D_{xy} + D_{yy} + H_1 + H_0), \quad (12)$$

where $G_0 = g_{00} \in K$, $H_1 = h_{10}D_x + h_{01}D_y \in K[D_x, D_y]$, $H_0 = h_{00} \in K$. Compare components of order 2 on both sides of equality (12), then we get a system of linear equations in coefficients h_{10}, h_{01}, g_{00} :

$$\begin{cases} a_{20} = h_{10}, \\ a_{11} = h_{01} + g_{00}, \\ a_{02} = g_{00}. \end{cases}$$

We find the unique solution of the system. Then, we compare coefficients in D_x on the both sides of (12), and so we get

$$h_{00} = a_{10} - a_{20}a_{02} - \partial_x(a_{20}).$$

Now we may compute a common obstacle as $P = L - (D_x + G_0) \circ (D_{xy} + D_{yy} + H_1 + H_0)$.

One may find the obstacle for type $(Y)(XX)$ analogously. \square

6.2 Three Factors

Here it is enough to consider the case of hyperbolic operators with symbol $XY(X + Y)$ and type $(X)(Y)(X + Y)$ of factorizations. In this case a common obstacle may be of orders 1 and 0 only (Theorem 4.19) and it is unique (Theorem 4.20).

Theorem 6.2. *Let*

$$L = D_x D_y (D_x + D_y) + a_{20} D_{xx} + a_{11} D_{xy} + a_{02} D_{yy} + a_{10} D_x + a_{01} D_y + a_{00},$$

where all $a_{ij} \in K$. The common obstacle of type $(X)(Y)(X + Y)$ is

$$\begin{aligned} \text{Obst}_{(X)(Y)(X+Y)} = & (a_{10} - a_{20}a_{11} + a_{20}^2 - \partial_x(a_{20}) + \partial_y(s_2))D_x + \\ & (a_{01} - a_{02}a_{11} + a_{02}^2 + \partial_x(-a_{11} + a_{02}))D_y + \\ & a_{00} + a_{20}a_{02}s_2 + s_2\partial_x(a_{20}) + (a_{20}\partial_x + \partial_{xy} + a_{02}\partial_y)(s_2), \end{aligned}$$

where $s_2 = a_{20} - a_{11} + a_{02}$.

Proof. Every factorization of type $(X)(Y)(X + Y)$ has the form:

$$L = (D_x + g_0) \circ (D_y + h_0) \circ (D_x + D_y + f_0). \quad (13)$$

Compare components of order 2 and get the only solution

$$h_0 = a_{20}, \quad g_0 = a_{02}, \quad f_0 = a_{11} - a_{02} - a_{20}.$$

Now, we may compute the common obstacle as the difference of the left and the right sides of the equation (13). \square

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